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Symmetries of the Lagrange equations and equivalent Lagrangians

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Abstract. The theoretical basis justifying the possibility of associating equivalent Lagrangians with (t, q) symmetries of Lagrangian systems of differential equations is given. The proof is based on well-known results and leads to a technique for finding equivalent Lagrangians that it is quite simple and more general than another recently proposed technique, since it is valid for systems of more than one differential equation and for both discrete and continuous families of symmetries.

1. Introduction

In a recent paper (Lutzky 1978) Lagrangian functions different from the standard one, $\dot{q}^2 - q^2$, were obtained out of monoparametric groups of pointlike transformations of symmetry (i.e. acting in the (t, q) space) of the differential equation

$$\ddot{q} + q = 0. \tag{1}$$

In this paper we justify geometrically the results obtained by Lutzky. We point out that these results are by no means restricted to systems with only one degree of freedom. In fact, as we shall see, our technique can be applied to Lagrangian systems of second-order differential equations of the kind

$$\ddot{q}_i = \alpha_i(t, q, \dot{q}), \qquad i = 1, \ldots, n. \tag{2}$$

On the other hand, Lutzky obtained monoparametric families of Lagrangian functions for equation (1) out of monoparametric families of symmetries of this equation. But, for generic systems of the kind (2) (and even when $n = 1$), it is known (González-Gascón 1977a) that, in general, they do not admit monoparametric families of pointlike transformations of symmetry. This is what happens, for instance, with the equation

$$\ddot{q} = q^2 + t^2. \tag{3}$$

For systems of the kind (2) not admitting monoparametric families of pointlike symmetries, the technique used by Lutzky cannot be applied. Nevertheless, as we shall explain, if (2) admits a discrete pointlike symmetry (for instance, in the case of (3) the symmetry $q \to -q$; $t \to -t$), the technique followed here is able to provide a recipe for obtaining Lagrangians equivalent to the original one. Our technique is also valid when (2) admits continuous groups of pointlike symmetries.

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2. The invariance in form of the Lagrange equations

It is well known (Pars 1965) that the Euler–Lagrange equations associated with a given function $L(t, q_i, \dot{q}_i)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_j}\right)-\frac{\partial L}{\partial q_j}=0, \qquad j=1, \ldots, n, \tag{4}$$

are a set of second-order differential equations possessing the following notable property: When they are subjected to the pointlike transformation

$$q_j=f_j(Q_1, \ldots, Q_n, t), \qquad j=1, \ldots, n, \tag{5}$$

then the new second-order differential equations for the Q variables are equivalent to (i.e. possess the same solutions as) the set of differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \hat{L}}{\partial \dot{Q}_j}\right)-\frac{\partial \hat{L}}{\partial Q_j}=0, \tag{6}$$

$\hat{L}$ being given by

$$\hat{L}(t, Q, \dot{Q})=L\left(t, f_i(Q_1, \ldots, Q_n, t), \sum_{j=1}^{n} \frac{\partial f_i}{\partial Q_j} \dot{Q}_j+\frac{\partial f_i}{\partial t}\right) \tag{7}$$

and $L(t, q_i, \dot{q}_i)$ being the Lagrangian function entering in (4).

Consider, for example, the pair of differential equations

$$\ddot{x}=0, \qquad \ddot{y}=0, \tag{4a}$$

x, y being the global Cartesian coordinates of the plane.

Under the coordinate change

$$x=r \cos \phi, \qquad y=r \sin \phi, \tag{5a}$$

the Lagrangian function L leading to $(4a)$,

$$L=\tfrac{1}{2}(\dot{x}^2+\dot{y}^2),$$

is transformed, according to (7), in the Lagrangian function

$$\hat{L}=\tfrac{1}{2}(\dot{r}^2+r^2\dot{\phi}^2),$$

and therefore the Euler–Lagrange equations corresponding to this $\hat{L}$ are

$$\ddot{r}-r\dot{\phi}^2=0, \qquad r^2\ddot{\phi}-2r\dot{r}\dot{\phi}=0. \tag{6a}$$

On the other hand, if we perform the change of variables $(5a)$ directly in the original equations $(4a)$, we immediately obtain the equations

$$\begin{aligned} &\ddot{r} \cos \phi-r(\sin \phi)\ddot{\phi}-2\dot{r}\dot{\phi} \sin \phi-r(\cos \phi)\dot{\phi}^2=0, \\ &\ddot{r} \sin \phi+r(\cos \phi)\ddot{\phi}-r(\sin \phi)\dot{\phi}^2+2\dot{r}\dot{\phi} \cos \phi=0, \end{aligned} \tag{8}$$

which can be rewritten as

$$(\ddot{r}, \ddot{\phi})\begin{pmatrix} \cos \phi & \sin \phi \\ -r \sin \phi & r \cos \phi \end{pmatrix}=[2\dot{r}\dot{\phi} \sin \phi+r(\cos \phi)\dot{\phi}^2, -2\dot{r}\dot{\phi} \cos \phi+r(\sin \phi)\dot{\phi}^2]. \tag{9}$$

Therefore under multiplication by the inverse of the matrix

$$\begin{pmatrix} \cos\phi & \sin\phi \\ -r\sin\phi & r\cos\phi \end{pmatrix},$$

we obtain

$$(\ddot{r}, \ddot{\phi}) = (2\dot{r}\dot{\phi}\sin\phi + r\dot{\phi}^2\cos\phi, -2\dot{r}\dot{\phi}\cos\phi + r\dot{\phi}^2\sin\phi)\begin{pmatrix} \cos\phi & -(\sin\phi)/r \\ \sin\phi & (\cos\phi)/r \end{pmatrix},$$

that is

$$(\ddot{r}, \ddot{\phi}) = (r\dot{\phi}^2, 2\dot{r}\dot{\phi}/r),$$

which are just equations ($6a$). This shows, in this case, that the transformed equations (8) are equivalent to (i.e. possess the same solutions as) equations ($6a$).

Let us return now to our general reasoning. Although the equivalence of equations (6) and the equations transformed from equations (4) under the change of variables (5) is well known, it is less well known (Gelfand and Fomin 1963, Hirsch 1898) that a completely similar result holds when equations (4) are transformed by a generic pointlike transformation in the (t, q) space:

$$q_i = f_i(Q_1, \dots, Q_n, T), \qquad t = f_0(Q_1, \dots, Q_n, T), \qquad i = 1, \dots, n. \tag{10}$$

In this case the transformed equations are equivalent to the Euler–Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}T}\left(\frac{\partial\hat{\hat{L}}}{\partial Q'}\right) - \frac{\partial\hat{\hat{L}}}{\partial Q} = 0, \tag{11}$$

where $\hat{\hat{L}}$ is now given by

$$\begin{aligned}\hat{\hat{L}} &= L\left(f_0, f_i, \frac{\mathrm{d}f_i/\mathrm{d}T}{\mathrm{d}t/\mathrm{d}T}\right)\frac{\mathrm{d}t}{\mathrm{d}T} \\ &= L\left(f_0, f_i, \frac{\sum_{j=1}^{n} \frac{\partial f_i}{\partial Q_j} Q_j' + \frac{\partial f_i}{\partial T}}{\sum_{j=1}^{n} \frac{\partial f_0}{\partial Q_j} Q_j' + \frac{\partial f_0}{\partial T}}\right)\left(\sum_{j=1}^{n} \frac{\partial f_0}{\partial Q_j} Q_j' + \frac{\partial f_0}{\partial T}\right), \\ Q_i' &= \mathrm{d}Q_i/\mathrm{d}T.\end{aligned} \tag{12}$$

We shall see in § 3 that this theoretical result will have important consequences when a pointlike symmetry of equations (4) is known.

3. The pointlike symmetry of the Lagrange equations and its consequences

Assume now that equations (4) are known to possess a pointlike symmetry of the kind (10), that is of the form

$$\bar{q}_i = f_i(q_1, \dots, q_n, t), \qquad \bar{t} = f_0(q_1, \dots, q_n, t). \tag{13}$$

In this case (by the definition of symmetry) the equations obtained by transforming (4) under (13) are equivalent to equations (4). On the other hand, the transformed equations are equivalent (see § 2) to equations (11), where $\hat{\hat{L}}$ is defined by (12).

Therefore, by the transitive property of the relations of equivalence, we can say that equations (4) and (11) are equivalent (i.e. they possess the same solutions). Therefore, and since in general it will not be possible to find a function $\Lambda(t, q)$ such that

$$\hat{L}(t, q, \dot{q}) = L(t, q, \dot{q}) = \mathrm{d}\Lambda(t, q)/\mathrm{d}t, \tag{14}$$

we obtain, in this way, and parting from any pointlike symmetry of (4), a Lagrangian system equivalent to the original one and, in general, not gauge equivalent to it.

Obviously if (4) admits not only a pointlike symmetry but a whole monoparametric (or multiparametric) group of symmetries of the kind (13), we would obtain (by means of (12)) a multiparametric set of Lagrangians $\hat{L}$ equivalent to L, some of them gauge equivalent to L (those for which (14) holds) and others not gauge equivalent to L.

In conclusion, a knowledge of the pointlike symmetries of a system of differential equations of the kind (2), when this system is equivalent to a Lagrangian system (Santilli 1978), is of great help in increasing the number of possible solutions of the so-called 'inverse problem', that is the problem of finding the Lagrangian functions, if any, whose associated Euler–Lagrange equations are equivalent to equations (2) The recipe for calculating the new Lagrangian is quite simple and is given by (12). The recipe is valid for both continuous and discrete symmetries.

This result is another indication of the great variety of applications that can be deduced from a knowledge of the symmetries of a set of differential equations (González-Gascón 1977a, b).

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